Math 4210 Tutorial 9

1.Assume that the normal random variables $X_1, X_2, \ldots, X_n$ of mean μ and variance σ^2 are uncorrelated. If $S_n = \frac{1}{n}\sum_{i=1}^n X_i$, find $E[S_n],\ Var(S_n)$.

Solution:

Recall: for $C_i \in \mathbb{R}$,

$$E\Big[\sum_{i=1}^n C_i X_i\Big] = \sum_{i=1}^n C_i \cdot E[X_i]$$

$$Var\Big(\sum_{i=1}^n C_i X_i\Big) = \sum_{i=1}^n C_i^2 \cdot Var(X_i) + 2\sum_{1\le i<j\le n} C_i C_j \cdot Cov(X_i, X_j)$$

Hence,

$$E[S_n] = E\Big[\frac{1}{n}\cdot\sum_{i=1}^n X_i\Big] = \sum_{i=1}^n \frac{1}{n}\cdot\mu = \mu.$$

$$Var(S_n) = \sum_{i=1}^n \frac{1}{n^2}\cdot Var(X_i) + 2\cdot\sum_{1\le i<j\le n} \frac{1}{n^2}\cdot Cov(X_i, X_j).$$

$$= \sum_{i=1}^n \frac{1}{n^2}\cdot\sigma^2$$

$$= \frac{\sigma^2}{n}.$$

2.If $X_1, X_2, \ldots, X_n$ are independent normal random variables with parameter μ_i, σ_i^2 ($X_i \sim N(\mu_i, \sigma_i^2)$), if $Y = \sum_{i=1}^n X_i$, then Y is still a normal random variable with parameters $(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$.

Solution:

Recall

if X is normal r.v. μ, σ^2 $\Longleftrightarrow$ Characteristic function $\varphi_\theta(X) = E[e^{i\theta X}]$

$= e^{i\theta\mu - \frac{\theta^2\sigma^2}{2}}$

Since $Y = \sum_{i=1}^{n} X_i$.

The Characteristic function of Y is

$$\varphi_Y(\theta) = E[e^{i\theta Y}] = E[e^{i\theta(\sum_{i=1}^{n} X_i)}]$$

$$= E[e^{i\theta X_1}] \cdot E[e^{i\theta X_2}] \cdots E[e^{i\theta X_n}]$$

$$= e^{i\theta\mu_1 - \frac{\theta^2\sigma_1^2}{2}} \cdot e^{i\theta\mu_2 - \frac{\theta^2\sigma_2^2}{2}} \cdots\cdots e^{i\theta\mu_n - \frac{\theta^2\sigma_n^2}{2}}$$

$$= e^{i\theta(\sum_{i=1}^{n}\mu_i) - \frac{\theta^2(\sum_{i=1}^{n}\sigma_i^2)}{2}}.$$

which is a characteristic function of a normal r.v. with parameter $(\sum\mu_i, \sum\sigma_i^2)$.

$$\Rightarrow Y \sim N\left(\sum_{i=1}^{n}\mu_i, \sum_{i=1}^{n}\sigma_i^2\right)$$

3.A random variable S is log-normal if

$$lnS \sim N(\mu, \sigma^2)$$

(a)Find the probability density function of S.
(b)If S(t) has log-normal distribution with drift r and volatility σ. Find $E[\frac{S(t+\Delta t)}{S(t)}]$, $E[(\frac{S(t+\Delta)}{S(t)})^2]$.

Solution:

$P(S \leq x)$. If $x \leq 0$. $P(S \leq x) = 0$.

If $x > 0$.

$$P(S \leq x) = P(\ln S \leq \ln x).$$

$$= P(Y \leq \ln x)$$

$$= \int_{-\infty}^{\ln x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

$$P_S(x) = \frac{d P(S \leq x)}{dx} = \frac{d \int_{-\infty}^{\ln x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy}{dx}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} \cdot \frac{d \ln x}{dx}$$

$$= \frac{1}{\sqrt{2\pi}\sigma \cdot x} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$$

(b)

$$\ln \frac{S(t+\Delta t)}{S(t)} \sim N\left((r-\frac{\sigma^2}{2})\Delta t,\ \sigma^2\Delta t\right)$$

$$Y_t = \frac{S(t+\Delta t)}{S(t)}$$

$$E[Y_t] = \int_{\mathbb{R}} y \cdot f_Y(y)\,dy$$

$$= \int_0^\infty y \cdot \frac{e^{-\frac{(\ln y-(r-\frac{\sigma^2}{2})\Delta t)^2}{2\sigma^2\Delta t}}}{\sqrt{2\pi}\,\sigma\sqrt{\Delta t}\,y}\,dy$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{\Delta t}} e^{-\frac{(\ln(y)-(r-\frac{\sigma^2}{2})\Delta t)^2}{2\sigma^2\Delta t}}\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{\Delta t}} e^{-\frac{z^2}{2\sigma^2\Delta t}}\, e^{(r-\frac{\sigma^2}{2})\Delta t} \cdot e^{z}\,dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{\Delta t}} e^{-\frac{z^2}{2\sigma^2\Delta t}+z+(r-\frac{\sigma^2}{2})\Delta t}\,dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{\Delta t}} e^{-\frac{(z-\sigma^2\Delta t)^2}{2\sigma^2\Delta t}} \cdot e^{r\Delta t}\,dz$$

$$= e^{r\Delta t}$$

$$E\left[\left(\frac{S(t+\Delta t)}{S(t)}\right)^2\right] = E[Y_t^2] = \int_0^\infty y^2 \cdot f_Y(y)\,dy$$

$$= \int_0^\infty y^2 \cdot \frac{1}{\sqrt{2\pi}\sigma\sqrt{\Delta t}\,y} e^{-\frac{\left(\ln y - (r-\frac{\sigma^2}{2})\Delta t\right)^2}{2\sigma^2 \Delta t}} dy$$

$$= e^{2r\Delta t + \sigma^2 \Delta t}$$

Alternatively,

since t is log-normal r.v.,

$$\ln Y_t = X. \quad X \sim N\left((r - \tfrac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t\right)$$

$$Y_t = e^X$$

$$E[Y_t] = E[e^X] = e^{(r-\frac{\sigma^2}{2})\Delta t + \frac{\sigma^2 \Delta t}{2}} = e^{r\Delta t}$$

Recall in Tutorial 7, $E[e^{\theta X}] = e^{\theta\mu + \frac{\theta^2\sigma^2}{2}}$

$X \sim N(\mu, \sigma^2)$

$$E[Y_t^2] = E[(e^X)^2] = E[e^{2X}] = e^{2\cdot(r-\frac{\sigma^2}{2})\Delta t + 2\cdot\sigma^2\Delta t}$$

$$= e^{2r\Delta t + \sigma^2\Delta t}$$